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Long strings and chiral non-singlets in matrix quantum mechanics

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Abstract: We study the non-singlet sectors of Matrix Quantum Mechanics in application to two-dimensional string theory. We use the chiral formalism, which operates directly with incoming and outgoing asymptotic states, related by a scattering operator. We argue that a general non-singlet scattering amplitude decomposes into scattering amplitudes in the adjoint. In the adjoint representation, we express the phase of the tree-level scattering amplitude as an integral over the Fermi sea. We claim that this expression holds for any, in general time-dependent, profile of the Fermi sea. In the case of stationary Fermi sea our formula reproduces Maldacena's scattering amplitude for a long string to go in and come back to infinity.

Keywords: Bosonic Strings, Long strings, Matrix Models, 2D Gravity.

Contents

1. Introduction

The two-dimensional string theory arises as the collective field theory of the gauged matrix quantum mechanics in "upside-down" gaussian potential, or shortly MQM [1].[1] MQM involves one gauge field A_i^j and one scalar field X_i^j, both hermitian $N \times N$ matrices. The theory is formally described by the action

$$\mathcal{S} = \int dt \, \mathrm{Tr} \left[P \, \nabla_A X - \tfrac{1}{2}(P^2 - X^2) \right], \tag{1.1}$$

where $\nabla_A X = \partial_t X - i[A, X]$ is the covariant time derivative. Since the potential is bottomless, one should introduce a cutoff and tune the size N with the cutoff before taking the large N limit.

The tachyon dynamics in 2D string theory with flat, linear dilaton, background is described by the *singlet sector* of the Hilbert space. In the singlet sector the gauge field simply plays the role of Lagrange multiplier assuring that the matrix coordinate and momentum can be simultaneously diagonalized. As a consequence, the action (1.1) describes a system of N non-relativistic free fermions in the upside-down quadratic potential. The ground state of the system is characterized by the Fermi level $E_F = -\mu$, where $g_s = 1/\mu$ is the string coupling constant.

The states obtained by introducing one or more Wilson lines in the adjoint representation belong to the *non-singlet sector* of the Hilbert space. The wave functions in this sector

[1] The relation between the 2D string theory and MQM was originally proposed by Kazakov and Migdal [2].

transform by non-trivial representations of SU(N) that can be obtained by tensoring the adjoint. Besides tachyons the non-singlet sector contains excitations which in the compactified Euclidean theory correspond to winding modes, or vortices, on the world sheet [3 – 5]. The non-singlet sector may contain more general string backgrounds. It was conjectured [6, 7] that adding two oppositely oriented Polyakov loops in the action (1.1),

$$\delta\mathcal{S} = \lambda \mathrm{Tr}\exp\left(\int_0^R dt A(t)\right) + \lambda \mathrm{Tr}\exp\left(-\int_0^R dt A(t)\right), \tag{1.2}$$

may deform the metric of the target space and produce Euclidean black hole background [8]. It is known that the deformations by winding modes are integrable [9] and the deformed partition function is a tau-function of a non-compact Toda hierarchy [7, 10] . However, knowing the partition function is not sufficient to reproduce the geometry of the background. In order to probe the background curvature, one should be able to calculate scattering amplitudes of tachyons in presence of a deformation by a non-singlet source. This problem is well posed in the Lorentzian formulation of the theory.

Recently, Maldacena [11] gave a world-sheet interpretation of the non-singlets in Lorentzian MQM. He considered mainly the states containing one Wilson line, which transform according to the simplest non-trivial representation, the adjoint. According to Maldacena, the wave function in the adjoint describes a particle-like excitation (impurity), interacting with the Fermi sea. The collective coordinate of this "adjoint particle", describes the position of the tip of a folded string that stretches from infinity. This interpretation passed several consistency tests. The tree level scattering amplitude for the tip of the long string, evaluated in [11] using the FZZ boundary Liouville amplitude [12], turned out to be the same as the one extracted from the wave function in the adjoint representation, $\mathcal{R}_{ij;kl}(\Omega) = \Omega_{ik}\mathcal{O}^*_{jl} - \delta_{ik}\delta_{jl}$ subsequently calculated in [13]. Furthermore, the density of states corresponding to the phase shift reproduced correctly the vortex-antivortex correlation function in the compactified Euclidean theory [11]. Further study of long strings in 2D string theory is presented in [14, 15].

In order to make the next steps towards constructing a solvable model of Lorentzian 2D black hole it is important to extend this picture to the whole non-singlet sector. As all allowed representations are contained in the tensor products of the adjoint, all non-singlet states can be described in terms of tachyons and adjoint particles, or long strings. The wave function of such a state can be in principle computed by solving the corresponding matrix Calogero equation. However, in spite of the fact that problem is integrable [16], this seems to be a rather hopeless task. Some interesting speculations concerning higher representations were presented in [17], but no quantitative description exists so far.

In this paper we argue that the scattering problem in the non-singlet sector of MQM can be solved by means of the *chiral formalism* introduced in [18] and later developed in [19 – 27, 14] , in which one performs a canonical transformation to the light cone variables

$$X_+ = \tfrac{X+P}{\sqrt{2}}, \qquad X_- = \tfrac{X-P}{\sqrt{2}}. \tag{1.3}$$

The wave functions in X_+ and X_- spaces describe the asymptotic states in the original theory respectively in the infinite past and future. The outgoing and the incoming states are related by Fourier transformation

$$\Phi^-(X_-) = \int dX_+ e^{i\mathrm{Tr}X_+X_-}\,\Phi^+(X_+), \tag{1.4}$$

which represents the scattering operator in the chiral basis. The Hamiltonian corresponding to the action (1.1) is bilinear in the new variables,

$$H = -\tfrac{1}{2}\mathrm{Tr}(X_+X_- + X_-X_+)\,, \tag{1.5}$$

and the general solution of the corresponding Schrödinger equation is

$$\Phi^\pm(X_\pm, t) = e^{\mp\frac{1}{2}N^2 t}\,\Phi^\pm(e^{\mp t}X_\pm). \tag{1.6}$$

In the non-singlet sector the wave function is a vector transforming according to a (not necessarily irreducible) representation $\mathcal{R}$ of SU(N):

$$\Phi_I^\pm(\Omega X_\pm \Omega^\dagger) = \sum_{J=1}^{\dim\mathcal{R}} \mathcal{R}_{I,J}(\Omega)\,\Phi_J^\pm(X_\pm), \qquad \Omega \in \mathrm{SU}(N), \tag{1.7}$$

where $\mathcal{R}_{I,J}(\Omega)$ is the matrix of the representation. The U(N) symmetry allows to reduce, as in the singlet sector, the original N^2 degrees of freedom to the N eigenvalues $x_1^\pm \dots x_N^\pm$ of the matrix X_+ or X_-.

Our claim is that one can construct a large N collective field theory for the scattering amplitude between the states Φ^- and Φ^+, which is given by the inner product

$$\left(\Phi^-|\Phi^+\right) = \int dX_+ dX_-\; e^{i\mathrm{Tr}X_+X_-} \sum_{J=1}^{\dim\mathcal{R}} \overline{\Phi_I^-(X_-)}\;\Phi_I^+(X_+). \tag{1.8}$$

This is achieved in two steps. The first step consists in integrating out the angular degrees of freedom in the matrix integration measure

$$dX_\pm = d\Omega_\pm\, dx_1^\pm \dots dx_N^\pm\, \Delta^2(X_\pm), \tag{1.9}$$

where $\Delta(X_\pm) = \prod_{i<j}(x_i^\pm - x_j^\pm)$ is the Vandermonde determinant. The second step is to formulate the result in terms of the collective field, the phase space eigenvalue density $\rho(x^+, x^-)$.

As a consequence of (1.7) , the angular dependence of the integrand in (1.8) is only through the factor

$$\sum_{K=1}^{\dim\mathcal{R}} \mathcal{R}_{I,K}(\Omega_+)\mathcal{R}_{K,J}(\Omega_-^\dagger) = \mathcal{R}_{I,J}(\Omega_+\Omega_-^\dagger).$$

The integral with respect to $\Omega = \Omega_+\Omega_-^\dagger$ depends only on the representation $\mathcal{R}$ and not on the wave functions:

$$I_{I,J}^{\mathcal{R}}(X_+, X_-) = \int_{\mathrm{SU}(N)} d\Omega\;\mathcal{R}_{I,J}(\Omega)\; e^{iX_+\Omega X_-\Omega^\dagger}. \tag{1.10}$$

In the singlet sector, $\mathcal{R} = \mathbb{I}$, this is Harish-Chandra-Itzykson-Zuber integral [28]. The general integral (1.10) has been first studied by Shatashvili [29] who found the complete solution, but in a form not explicitly invariant under permutations of eigenvalues and hence not immediately applicable in the large N limit. A nice symmetric formula for the adjoint representation, $\mathcal{R}_{ik,jl}(\Omega) = \Omega_{ij}\Omega^\dagger_{lk} - \Omega_{ik}\Omega^\dagger_{lj}$ was originally proposed by Morozov [30] and subsequently proved by Eynard and collaborators [31, 32]. Generalization of Morozov-Eynard formula for any representation was found in [33]. Therefore the first step, the $\mathrm{U}(N)$ integration, is essentially accomplished.

In order to extract the large N limit and construct the collective field theory, we need to evaluate the inner product (1.8) for the wave functions that describe low energy excitations above the ground state. The latter is constructed by to filling the fermionic energy levels of the singlet sector up to the the Fermi level $E_F = -\mu$. It is sufficient to solve the problem for a standard set of functions spanning the sector containing p adjoint particles,

$$\hat{\Phi}^{\pm}_{i_1\cdots i_p, j_1\cdots j_p}(\xi^\pm_1, \ldots, \xi^\pm_p; X_\pm) = \left[\frac{1}{\xi^\pm_1 + X_\pm}\right]_{i_1 j_{\sigma^\pm(1)}} \cdots \left[\frac{1}{\xi^\pm_p + X_\pm}\right]_{i_p j_{\sigma^\pm(p)}} \Phi^\pm_0(X_\pm), \quad (1.11)$$

where $\Phi^\pm_0$ is the ground state wave function and $\sigma^\pm$ are permutations from S_N. Here we suppressed the time dependence, which can be reconstructed from (1.6). The functions (1.11) transform according to the tensor product of p adjoints. The projection to the irreducible components of the tensor product is done by summing over the permutations with a character of the symmetric group.

The integral for the inner product of functions of the type (1.11) is similar to that for the mixed correlators in the two-matrix model, which were evaluated in the large N limit by Eynard and Orantin [34]. The authors of [34] found that a generic mixed correlator decomposes as a sum of products of two-point mixed correlators. An important for us fact is that the coefficients in the sum do not depend on the radial part of the matrix measure. Therefore the result of [34] must be applicable also in the case of the inverse matrix oscillator, which means that any scattering process involving only adjoint particles can be decomposed into one-particle scattering amplitudes.

In this way the success of the chiral approach is guaranteed if it works for the adjoint representation. Our aim here is to demonstrate that this is indeed the case. We will obtain a formula for the scattering amplitude of the adjoint particle in terms of an integral over the Fermi sea, which we expect to be valid for any, in general time-dependent, tachyon background. In the case of stationary Fermi sea we recover Maldacena's expression [11] for the scattering amplitude for a long string to come in and go back to infinity. Our more general answer can be used to calculate the tree-level processes involving one adjoint particle and any number of tachyons. Scattering amplitudes involving several adjoint particles will be considered in a future publication [35].

2. The scattering amplitude in the adjoint representation

It will be convenient to absorb into the wave function a Vandermond determinant from the measure (1.9):

$$\Psi^{\pm}(X_{\pm}) := \Delta(X_{\pm})\, \Phi^{\pm}(X_{\pm}). \tag{2.1}$$

Then a complete set of wave functions in the adjoint sector is given by[2]

$$\hat{\Psi}^{\pm}_{ij}(\xi^{\pm}; X_{\pm}) = \left[\frac{1}{\xi^{\pm} + X_{\pm}}\right]_{ij} \det_{kl}\left[\psi^{\pm}_{E^{\pm}_k}(x^{\pm}_l)\right]. \tag{2.2}$$

The last factor is the general eigenfunction of the matrix Hamiltonian in the singlet sector, which is a Slater determinant of one-fermion eigenfunctions

$$\psi^{\pm}_{E}(x_{\pm}) \;=\; \tfrac{1}{\sqrt{2\pi}}\,(x_{\pm})^{\pm iE - \frac{1}{2}}. \tag{2.3}$$

We are going to consider only states which represent incoming leftmovers and outcoming rightmovers. Then the eigenvalues of $X_{\pm}$ can be assumed positive and the operator functions (2.2) are analytic in the complex ξ-plane cut along the negative real axis. One can think of ξ^{+} and ξ^{-} as the phase space coordinates of the adjoint particle. The Hamiltonian (1.5) acts on the wave function (2.2) as

$$H\hat{\Psi}^{\pm} = \mp i\sum_{k}\left(x^{\pm}_{k}\tfrac{\partial}{\partial x^{\pm}_{k}} + \tfrac{1}{2}\right)\Psi^{\pm} = \left[\pm i\left(\xi^{\pm}\tfrac{\partial}{\partial\xi^{\pm}} + 1\right) + E^{\pm}_{1} + \ldots + E^{\pm}_{N}\right]\hat{\Psi}^{\pm}. \tag{2.4}$$

We will be eventually interested in evaluating the inner product for the eigenfunctions of the Hamiltonian,

$$\Psi^{\pm}_{\tilde{E}^{\pm}}(X_{\pm}) = (X_{\pm})^{\pm i\tilde{E}^{\pm}} \det_{kl}\left[\psi^{\pm}_{E^{\pm}_k}(x^{\pm}_l)\right], \tag{2.5}$$

which are obtained as Mellin transforms of (2.2):

$$\Psi^{\pm}_{\tilde{E}^{\pm}}(X_{\pm}) = \pm\tfrac{i\sinh\pi\tilde{E}}{\pi}\int_0^{\infty} d\xi\; \xi^{\pm i\tilde{E}^{\pm}}\;\Psi^{\pm}(\xi^{\pm}; X_{\pm}). \tag{2.6}$$

Our aim is to evaluate the scattering amplitude for the adjoint particle in the large N limit, when the eigenvalues can be replaced by a collective mean field. For that it is sufficient to consider the diagonal elements of the scattering operator, $E^{\pm}_{k} = E_k,\ k = 1, \ldots, N$, which implies $\tilde{E}^{+} = \tilde{E}^{-}$. The non-diagonal elements, which are of subleading order, allow to study the back reaction of the adjoint particle to the collective field.

We will proceed as follows. First we perform the angular integration in the inner product of the functions (2.2) using Morozov-Eynard formula. Then we consider the large N limit, in which the result takes the form of Fredholm determinant, and evaluate the leading contribution at large cosmological constant μ. Finally we perform the Mellin transform according to (2.6) and obtain the reflection factor as a function of the energy $\tilde{E}$.

2.1 Eliminating the angles

The angular integration in the inner product of the wave functions (2.2) is evaluated by Morozov-Eynard formula [32]

$$\int\limits_{\mathrm{SU}(N)} d\Omega\,\mathrm{Tr}\left(\tfrac{1}{\xi_{-}+X_{-}}\Omega\tfrac{1}{\xi_{+}+X_{+}}\Omega^{\dagger}\right) e^{\mathrm{Tr}(iX_{-}\Omega X_{+}\Omega^{\dagger})} = \frac{\det\left(\hat{S} + i\frac{1}{\xi_{-}+X_{-}}\hat{S}\frac{1}{\xi_{+}+X_{+}}\right) - \det\hat{S}}{i\,\Delta(X_{-})\,\Delta(X_{+})}, \tag{2.7}$$

[2]Since we do not subtract the trace, this function has a small component in the singlet sector.

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where $\hat{S}$ is the matrix with matrix elements $S_{jk} = e^{ix_j^+ x_k^-}$ and $X_\pm = \mathrm{diag}(x_1^\pm, \ldots x_N^\pm)$. Plugging (2.7) in the definition of the inner product (1.8), we get

$$(\hat{\Psi}^-|\hat{\Psi}^+) = -i\int_0^\infty d^N x^+ d^N x^- \overline{\det_{kl}\left[\psi^-_{E_k}(x_l^-)\right]}\,\det_{kl}\left[\psi^+_{E_k}(x_l^+)\right]\times\det\left[\hat{S} + i\tfrac{1}{\xi_- + X_-}\hat{S}\tfrac{1}{\xi_+ + X_+}\right]. \tag{2.8}$$

Here we dropped the second term in the Morozov-Eynard formula, which is needed to cancel the constant term in the expansion of the r.h.s. at $\xi_\pm \to \infty$. This term does not depend on $\xi_\pm$ and therefore is irrelevant for the scattering phase. Note that the Vandermonde factors from the measure disappear due to the redefinition (2.1) of the wave function.

A form of the inner product suitable for taking the large N limit is obtained if we rewrite (2.8) as a determinant of double integrals:

$$(\hat{\Psi}^-|\hat{\Psi}^+) = R^{\mathrm{ad}}(\xi_+, \xi_-)\prod_{k=1}^N R(E_k), \tag{2.9}$$

$$R^{\mathrm{ad}}(\xi_+, \xi_-) = -i\det_{jk}\left[\delta(E_j - E_k) + iK(E_j, E_k)\right], \tag{2.10}$$

$$K(E', E) = \langle E'|\frac{1}{(\xi_+ + x_+)(\xi_- + x_-)}|E\rangle. \tag{2.11}$$

Here $R(E)$ denotes the fermion reflection coefficient, or the bounce factor, determined by the inner product of the one-particle eigenfunctions (2.3),

$$\int_0^\infty dx_+ dx_- \overline{\psi_-^{E'}(x^-)}\, e^{ix_+ x_-}\psi_+^{E'}(x^+) = R(E)\,\delta(E - E'), \tag{2.12}$$

and the r.h.s. of (2.11) is defined as

$$\langle\, E'|f|E\,\rangle := \frac{\int_0^\infty dx_+ dx_- \overline{\psi_-^{E'}(x^-)}\, f(x_-, x_+)\, e^{ix_+ x_-}\psi_-^{E'}(x^+)}{\sqrt{R(E)\,R(E')}}. \tag{2.13}$$

In particular,

$$\langle\, E'|1|E\,\rangle = \delta(E' - E). \tag{2.14}$$

The second factor in the inner product (2.8) is the inner product of the singlet states whose wave functions are given by the Slater determinants in (2.2). Therefore the first factor, $R^{\mathrm{ad}}(\xi_+, \xi_-)$, gives the scattering amplitude (more precisely, the reflection coefficient) for the adjoint particle as a function of the collective coordinates, ξ_+ and ξ_-, of the initial and final states.

2.2 The scattering amplitude as a Fredholm determinant

The expression (2.10) for the inner product was derived for a generic wave function of the form (2.2). Now we will specify the Slater determinant in (2.2) to be the wave function of the Fermi sea filled up to $E_F = -\mu$. For this purpose we introduce a large cutoff $\Lambda \gg \mu$, say, by putting a wall at $x_\pm \sim \sqrt{\Lambda}$, so that the spectrum becomes discrete. Then the density of the energy levels in the singlet sector is given by [18, 19]

$$\rho(E) = \frac{\log\Lambda}{2\pi} - \frac{1}{2\pi}\frac{d\phi_0(E)}{dE}, \qquad \phi_0(E) = -i\log R(E), \tag{2.15}$$

and the level spacing is

$$\Delta E = \frac{2\pi}{\rho(E)} \approx \frac{1}{\log\Lambda}. \tag{2.16}$$

The total number N of eigenvalues is given by the number of energy levels between the bottom of the regularized Fermi sea, $E_{\rm cutoff} = -\Lambda$, and the Fermi level $E_F = -\mu$:

$$N = \int_{-\Lambda}^{-\mu} dE\,\rho(E). \tag{2.17}$$

Finally, the inner product (2.12) is replaced by the following relation for the regularized eigenfunctions:

$$\int_0^\infty dx_+ dx_- \overline{\psi_-^{E_j}}\, e^{ix_+x_-} \psi_+^{E_k} = R(E_j)\,\rho(E_j)\,\delta_{jk}\,. \tag{2.18}$$

We would like to evaluate the scattering amplitude (2.10) for the case where $E_1, \dots, E_N$ are the allowed energy levels in the interval $[-\Lambda, -\mu]$. In presence of a cutoff, the expression (2.10) takes the form

$$\begin{aligned} R^{\rm ad}(\xi_+,\xi_-) &= -i\det_{jk}\left[\rho(E_j)\delta_{jk} + iK(E_j,E_k)\right] \\ &= -i\prod_{k=1}^{N}\rho(E_k)\times\det_{jk}\left[\delta_{jk} + i\tfrac{\Delta E_j}{2\pi}\;K(E_j,E_k)\right]. \end{aligned} \tag{2.19}$$

The first factor is an irrelevant infinite constant and will be neglected. The second factor has a smooth limit $\Lambda\to\infty$:

$$\begin{aligned} R^{\rm ad}(\xi_+,\xi_-) &= 1 + i\sum_j \tfrac{\Delta E_j}{2\pi}K(E_j,E_j) - \frac{1}{2!}\sum_{j,k}\tfrac{\Delta E_j}{2\pi}\tfrac{\Delta E_k}{2\pi}\left|\begin{pmatrix} K(E_j,E_j) & K(E_j,E_k) \\ K(E_k,E_j) & K(E_k,E_k)\end{pmatrix}\right| + \cdots \\ &\to 1 + i\int_{-\infty}^{-\mu}\tfrac{dE}{2\pi}K(E,E) - \frac{1}{2!}\int_{-\infty}^{-\mu}\tfrac{dE}{2\pi}\int_{-\infty}^{-\mu}\tfrac{dE'}{2\pi}\left|\begin{pmatrix} K(E,E) & K(E,E') \\ K(E',E) & K(E',E')\end{pmatrix}\right| + \cdots. \end{aligned} \tag{2.20}$$

The series is by definition the Fredholm determinant of the kernel (2.11), restricted to the interval $[-\infty, -\mu]$:

$$R^{\rm ad}(\xi_+,\xi_-) = {\rm Det}[1 + iK]. \tag{2.21}$$

Therefore the scattering phase $S(\xi_+,\xi_-)$, defined by

$$R^{\rm ad}(\xi_+,\xi_-) = \exp\left[iS(\xi_+,\xi_-)\right], \tag{2.22}$$

has the following integral representation:

$$S(\xi_+,\xi_-) = \sum_{n=1}^{\infty}\frac{(-i)^{n-1}}{n}{\rm Tr}\,K^n, \tag{2.23}$$

$${\rm Tr}\,K^n := \int_{-\infty}^{-\mu}\tfrac{dE_1}{2\pi}\cdots\tfrac{dE_n}{2\pi}\;K(E_1,E_2)\cdots K(E_n,E_1). \tag{2.24}$$

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2.3 Tree level calculation of the reflection factor

The Fredholm kernel $K(E, E')$ can be evaluated semiclassically for large negative energies. The easy calculation is given in appendix A. The result is

$$K(E+\epsilon, E-\epsilon) = \left(\frac{\xi_+}{\xi_-}\right)^{i\epsilon} \frac{\left(-\frac{\xi_+\xi_-}{E}\right)^{i\epsilon} - \left(-\frac{\xi_+\xi_-}{E}\right)^{-i\epsilon}}{i\sinh(2\pi\epsilon)} \frac{\pi}{\xi_+\xi_- + E}\,. \tag{2.25}$$

The corrections are of order $1/E$. It is obvious that the integral for $\mathrm{Tr}K^n$ is convergent for $n > 1$ and behaves as $o(\mu^{1-n})$. Therefore at tree level the phase (2.23) is given by the first term of the series,

$$S(\xi_+, \xi_-) = \mathrm{Tr}K = \int_{-\tilde\Lambda}^{-\mu} dE\ K(E, E), \tag{2.26}$$

$$K(E, E) = \frac{\ln(\xi_+\xi_-) - \ln(-E)}{\xi_+\xi_- + E}. \tag{2.27}$$

This integral is logarithmically divergent at minus infinity. To make it convergent, we cut it off at $E = -\tilde\Lambda$, where $\mu \ll \tilde\Lambda \ll \Lambda$. The new cutoff $\tilde\Lambda$ will be given below a precise meaning in terms of the world-sheet theory. Let us introduce the variable

$$s = \tfrac{1}{2}\log(\xi_+\xi_-/\mu) \tag{2.28}$$

and the function (it already appeared in Maldacena's paper)

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{x} d\zeta \left(\frac{\zeta}{\tanh\zeta} + \zeta\right). \tag{2.29}$$

Then we can write the regularized integral (2.26) as

$$S(\xi_+, \xi_-) = f\left(\tfrac{1}{2}\log(\tilde\Lambda/\mu) - s\right) - f(-s) \approx \frac{1}{\pi}\left(\tfrac{1}{2}\log(\tilde\Lambda/\mu) - s\right)^2 - f(-s)\,. \tag{2.30}$$

In the second line we used the asymptotics $f(x) \approx x^2/\pi$ at $x \gg 1$.

The leading corrections to this tree-level formula come from the terms of order $1/\mu$ in the semi-classical expression for the kernel (2.27), as well as from the second term, $\mathrm{Tr}K^2$, of the series (2.23). It however natural to expect that the $1/\mu$ corrections cancel and the perturbation series for the scattering phase is in $1/\mu^2$.

2.4 The scattering phase as an integral over the Fermi sea

In the derivation of the scattering phase we did not use the particular form (2.3) the one-particle functions. Therefore the derivation remains valid also in presence of a tachyon source, whose only effect is that the one-particle wave functions get deformed [18] . In this case the scattering phase will be given by the same expression (2.23), but with a deformed kernel K. The leading term, the trace (2.26), can be expressed alternatively as an integral over the Fermi sea,

$$S(\xi_+, \xi_-) = \int_0^\infty \frac{dx_+ dx_-}{2\pi} \frac{\rho(x_+, x_-)}{(\xi_+ + x_+)(\xi_- + x_-)}\,, \tag{2.31}$$

where $\rho(x_+, x_-)$ is the semi-classical density of the fermionic liquid. The expression (2.31) can be useful if we want to calculate the scattering phase in more general, time-dependent, tachyon backgrounds. In the case of a stationary Fermi sea the density is given by

$$\rho(x_+, x_-) = \begin{cases} 1 & \text{if } \mu < x_+ x_- < \tilde{\Lambda}; \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

2.5 Evaluation of the scattering phase in the energy space

The scattering amplitude for the eigenstates with given energy (2.5) is obtained by applying the integral transformation (2.6) to both arguments of $R^{\mathrm{ad}}(\xi_+, \xi_-)$:

$$\tilde{R}^{\mathrm{ad}}(\tilde{E}_+, \tilde{E}_-) = \tfrac{1}{\pi^2}\sinh(\pi\tilde{E}_+)\sinh(\pi\tilde{E}_-)\int_0^\infty d\xi_+ d\xi_- \xi_+^{i\tilde{E}_+}\xi_-^{i\tilde{E}_-}\; e^{iS(\xi_+,\xi_-)}. \tag{2.33}$$

For a stationary Fermi sea the r.h.s. depends on ξ_+ and ξ_- only through the variable s defined in (2.28), hence the l.h.s. contains a delta function imposing the energy conservation:

$$\begin{aligned} \tilde{R}^{\mathrm{ad}}(\tilde{E}_+, \tilde{E}_-) &= e^{-i\delta^{\mathrm{ad}}(\tilde{E}_+)}\; \delta(\tilde{E}_+ - \tilde{E}_-); e^{-i\delta^{\mathrm{ad}}(\tilde{E})} \\ &= \tfrac{1}{4\pi^2}\, e^{2\pi\tilde{E}}\; \mu^{i\tilde{E}+1} \int_{-\infty}^{\infty} ds\, e^{2s(i\tilde{E}+1)+i\frac{1}{\pi}(s-\frac{1}{2}\log\frac{\tilde{\Lambda}}{\mu})^2 - if(-s)}. \end{aligned} \tag{2.34}$$

In the last equation we replaced the sine function in front of the integral by exponent, which is justified for large positive energies, $\tilde{E} \sim \frac{1}{2\pi}\log(\tilde{\Lambda}/\mu)$.

We subtract, as in [11], the energy gap that separates the singlet and the adjoint sectors and introduce the shifted energy[3]

$$\hat{\epsilon} = \tilde{E} + \frac{1}{2\pi}\ln\frac{\mu}{\tilde{\Lambda}}. \tag{2.35}$$

Then we write the integral (2.34) in the form

$$e^{-i\delta^{\mathrm{ad}}(\tilde{E})} = \tfrac{1}{4\pi^2}\,\tilde{\Lambda}\;\mu^{i\tilde{E}}e^{2\pi\hat{\epsilon}}\; e^{\frac{i}{4\pi}(\log\frac{\tilde{\Lambda}}{\mu})^2}\int_{-\infty}^{\infty} ds\, e^{2s(i\hat{\epsilon}+1)+if(s)-i\pi/6}, \tag{2.36}$$

where we used the property $f(-x) + f(x) = \frac{x^2}{\pi} + \frac{\pi}{6}$. The integral can be taken exactly using the remarkable fact (see appendix B) that the function $e^{2s+if(s)}$ reproduces itself after Fourier transformation. Thus we obtain for the phase factor

$$e^{-i\delta^{\mathrm{ad}}(\tilde{E})} \sim \tilde{\Lambda}\,\mu^{i\tilde{E}}e^{\frac{i}{\pi}(\frac{1}{2\pi}\log\frac{\tilde{\Lambda}}{\mu})^2}e^{-if(-\pi\hat{\epsilon})-i\pi/6} = \tilde{\Lambda}\,e^{i\pi(\frac{1}{2\pi}\log\tilde{\Lambda})^2}\times e^{-i\pi(\tilde{E}-\frac{1}{2\pi}\log\tilde{\Lambda})^2-i\pi}\times e^{if(\pi\hat{\epsilon})}. \tag{2.37}$$

We get for the scattering phase, neglecting the large cutoff-dependent constant,

$$\delta^{\mathrm{ad}}(\tilde{E}) = \pi(\tilde{E} - \tfrac{1}{2\pi}\log\tilde{\Lambda})^2 - f(\pi\hat{\epsilon}). \tag{2.38}$$

The answer is in accord with Maldacena's calculation of the scattering phase based on the FZZ formula for the boundary two-point function in Liouville theory, eq. (A.4) in appendix A of [11]. Comparing the quadratic cutoff-dependent factors, we see that the cutoff $\tilde{\Lambda}$ in the integral over energies should be identified with the square of the large boundary cosmological constant of the FZZT brane, $\tilde{\Lambda} = \mu_B^2$.

[3] Note that our energy variable $\tilde{E}$ differs from the variable ϵ used in [11] by a cutoff-dependent constant, $\tilde{E} = \epsilon + \frac{1}{2\pi}\log\tilde{\Lambda}$.

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3. Discussion

In this paper, we explained how to use the chiral formalism of MQM in order to evaluate the scattering amplitudes in the non-singlet sector. The scattering operator is given by matrix Fourier transformation and its matrix elements can be in principle evaluated by applying recently discovered generalizations of the Harish-Chandra-Itzykson-Zuber formula. We considered in detail the scattering process in which the incoming and the outgoing states are in the adjoint representation. The wave function in the adjoint describes a particle-like excitation interacting with the Fermi sea, which was identified in [11] as a long folded string stretching from infinity.

We showed that the expression for the scattering amplitude such an 'adjoint particle' can be written in the form of a Fredholm determinant. The Fredholm kernel depends on the profile of the Fermi sea of the singlet sector. In the case of stationary Fermi surface, $x^+x^- = \mu$, we reproduced Maldacena's phase shift for a single long string [11]. Our tree-level formula (2.31) for the scattering phase as an integral over the Fermi sea is actually valid also for time-dependent perturbations of the Fermi surface, e.g., by a tachyon source. This more general expression can be used to evaluate the amplitudes of emission or absorption of tachyons by the long string.

The scattering amplitudes involving p adjoint particles are given by the inner products of the wave functions (1.11). In the case when $(\sigma^+)^{-1}\sigma^- = 1$, the tree-level amplitude factorizes into p one-particle amplitudes. In the general case, when $(\sigma^+)^{-1}\sigma^-$ is a permutation with n cycles, the scattering amplitude factorizes into a product of one-cycle amplitudes. For each such amplitude one can use the result of [34] to express it as a sum of products of one-particle amplitudes:

$$R(\xi_1^+, \dots, \xi_k^+; \xi_1^-, \dots, \xi_k^-) = \sum_{\sigma \in S_k} C_\sigma^{(k)}(\xi_1^+, \xi_1^-, \dots, \xi_k^+, \xi_k^-) \prod_{i=1}^{k} R^{\mathrm{ad}}(\xi_i^+, \xi_{\sigma(i)}^-). \tag{3.1}$$

In this formula all the dependence on the eigenvalue distribution is contained in the one-particle amplitudes, while the coefficients are universal and related only to the $\mathrm{U}(N)$ integration. For example, the one-cycle-of-length-two amplitude is given by

$$R(\xi_1^+, \xi_2^+; \xi_1^-, \xi_2^-) = \frac{R^{\mathrm{ad}}(\xi_1^+, \xi_1^-) R^{\mathrm{ad}}(\xi_2^+, \xi_2^-) - R^{\mathrm{ad}}(\xi_1^+, \xi_2^-) R^{\mathrm{ad}}(\xi_2^+, \xi_1^-)}{(\xi_1^+ - \xi_2^+)(\xi_1^- - \xi_1^-)}. \tag{3.2}$$

One can evaluate in this way the most general scattering process involving any number of tachyons and adjoint particles. The corresponding asymptotic states are obtained by replacing in (1.11) the last factor, the ground state singlet wave function, with an excited singlet state.

An alternative approach to study the scattering in the non-singlet sector consists in diagonalizing the scattering matrix given by the inner product (1.8). This is possible because the problem is integrable. In the case of the "upside-up" matrix oscillator such an approach, based on a spectrum-generating algebra that generalizes W_∞ of the singlet sector, was developed recently by Y. Hatsuda and Y. Matsuo [36]. In the "upside-down"

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case the generators of this algebra create non-singlet discrete states[4] and can be used to write down Ward identities for the S-matrix elements.

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A. Quasiclassical calculation of the integration kernel

Consider first the simplest problem, the calculation of the diagonal $E' = E$ of the kernel (2.11) for large negative energy, $-E \gg 1$. Introduce the parametrization

$$x_\pm = \sqrt{r}\, e^{\pm\tau}, \qquad \xi_\pm = \sqrt{\rho}\, e^{\pm\sigma} \tag{A.1}$$

and evaluate the τ-integral in the numerator of (2.13):

$$\begin{aligned}&\textstyle\int_0^\infty dx_+ dx_- (x_+ x_-)^{iE-1/2} \frac{e^{ix_+ x_-}}{(\xi_+ + x_+)(\xi_- + x_-)} \\ &= \textstyle\int_{-\infty}^\infty d\tau \int_0^\infty \frac{dr}{\sqrt{r}} r^{iE} e^{ir} \frac{1}{r+\rho+2\sqrt{r\rho}\cosh(\tau-\sigma)} = \displaystyle\int_0^\infty \frac{dr}{\sqrt{r}} r^{iE} e^{ir} \frac{\ln r - \ln\rho}{r-\rho}.\end{aligned} \tag{A.2}$$

The remaining integral in

$$K(E,E) = \frac{\int_0^\infty \frac{dr}{\sqrt{r}} r^{iE} e^{ir} \frac{\ln r - \ln\rho}{r-\rho}}{\int_0^\infty \frac{dr}{\sqrt{r}} r^{iE} e^{ir}} \tag{A.3}$$

can be evaluated semiclassically. Up to $o(\frac{1}{E})$ correction the integrals in the numerator and in the denominator are saturated by the same saddle point, $r = -E$. Therefore in the leading order

$$K(E,E) = \frac{\ln(-E) - \ln\rho}{-E-\rho} = \frac{\ln(-E) - \ln(\xi_+\xi_-)}{-E - \xi_+\xi_-}. \tag{A.4}$$

The off-diagonal elements $K(E,E')$ are evaluated similarly. The τ-integral in this case gives

$$\int_{-\infty}^\infty d\tau \frac{e^{2i\epsilon\tau}}{r+\rho+2\sqrt{r\rho}\cosh(\tau-\sigma)} = \frac{\pi}{r-\rho} \frac{(r/\rho)^{i\epsilon} - (r/\rho)^{-i\epsilon}}{i\sinh 2\pi\epsilon}\, e^{2i\epsilon\sigma} \tag{A.5}$$

and the saddle point approximation of the integral in r yields (2.25) .

[4] We recommend [37] as a good review paper about the role of the W_∞ symmetry and the discrete states in the singlet sector of MQM.

B. Propertirs of the function (2.29)

1. Symmetry:

$$f(-x)+f(x)=\frac{x^2}{\pi}+\frac{\pi}{6}\,. \tag{B.1}$$

2. Series expansion at $x \to +\infty$:

$$f(x)=\frac{\pi^2}{6}+\frac{x^2}{\pi}-\sum_{n\geq 1}\left(\frac{x}{n}+\frac{1}{n^2}\right)e^{-2nx}. \tag{B.2}$$

3. Functional identity:

$$\left(e^{-i\pi\partial_x}+e^{2x}\right)e^{if(x)}=e^{if(x)}, \tag{B.3}$$

4. Fourier transform:

$$\int_{-\infty}^{\infty} ds\, e^{2s}\, e^{2is\hat{\epsilon}+if(s)}=ie^{-2\pi\hat{\epsilon}}\, e^{-if(-\pi\hat{\epsilon})}. \tag{B.4}$$

The symmetry property (B.1) and the series expansion (B.2) follow directly from the definition (2.29). The functional equation is equivalent to

$$f(x-i\pi)-f(x)=-i\int_{-\infty}^{x} dx(\coth(x)+1)=-i\log(1-e^{2x}). \tag{B.5}$$

We do not have a complete analytic proof for the fourier transform (B.4) , so we had to complete our analytic argument with numerical evaluation. Equation (B.3) is symmetric under Fourier transformation

$$e^{if(x)}=\frac{1}{2\pi}\int_{\mathbb{R}} dy\, e^{-\frac{2}{\pi}ipx+i\tilde{f}(y)},\quad e^{i\tilde{f}(y)}=\int_{\mathbb{R}} d\hat{s}\, e^{\frac{2}{\pi}ipx+if(x)}.$$

Therefore the Fourier image satisfies the same functional equation. There are two obvious solutions, $\tilde{f}(p)=f(p)$ and $\tilde{f}(p)=-f(-p)$, up to a periodic function under $p\to p+i\pi$. The saddle point evaluation of the integral, valid for large negative p, is compatible with the second choice,

$$\tilde{f}(p)=-f(-p)-i\ln h(p),$$

where h is periodic, $h(p+i\pi)=h(p)$. Further, the function $h(p)$ must vanish exponentially at $p\to-\infty$, where the quasiclassics holds and we know that $\tilde{f}(p)\sim-\frac{p^2}{\pi}$. Therefore h is expanded at $p\to-\infty$ as a power series in e^{2p}. Finally, since $f(x)$ vanishes exponentially for large negative x, the function $h(p)$ must have a simple pole $\frac{\pi}{2ip}$ at $p=0$. These condition fix the form of function h as

$$h(p)=-i\pi\frac{1+h_1e^{2p}+h_2e^{4p}+\cdots}{e^{2p}-1}\,.$$

The numerical evaluation of the integral (we thank D. Volin for the help with that) is compatible with $h_1 = h_2 = \ldots = 0$. Therefore

$$h(p) = \frac{i}{e^{2p} - 1}, \tag{B.6}$$

which implies

$$e^{i\tilde{f}(p)} := \int_{\mathbb{R}} dx\, e^{\frac{2}{\pi} ipx + if(x)} = -i\pi \frac{e^{-if(-p)}}{e^{2p} - 1}. \tag{B.7}$$

Applying the shift equation (B.5) we get (B.4).

Equation (B.4) is actually a particular case of the formula for the Fourier transformation of the quantum dilogarithm presented in the Appendix of ref. [38].

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